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A comparison of inertial stability conditions in the planetary semi-geostrophic and quasi-equilibrium models

by
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Abstract

This report derives inertial stability conditions for the planetary semi-geostrophic (PSG) and quasi-equilibrium (QE) models which can be tested against observational data. The process required to extract suitable data from archive is explained, and some comparisons are made. It is shown to be virtually impossible to find data which can be used to verify the PSG stability condition, and hence the applicability of the PSG model itself. The QE stability condition is compared with observational data, and is shown, on rather limited evidence, to be satisfied in the majority of cases.

1 Introduction

The semi-geostrophic models are a class of models of atmospheric flow governed by balanced equations. They are derived from the hydrostatic primitive equations by making various assumptions. The models can be classified as lying between the primitive equations, and the Phillips type II quasi-geostrophic equations because of the nature of the approximations made: the semi-geostrophic models make less restrictive assumptions than the quasi-geostrophic models.

Here we will look at two semi-geostrophic models: firstly, the planetary semi-geostrophic equations (PSG) developed by Shutts (see [2]), which are a set of balanced equations derived using Hamilton's principle, and applied to a planetary (i.e. spherical) domain, and secondly, the quasi-equilibrium model (QE) developed by Hoskins (see [1]). We will investigate what conditions each model predicts as being necessary for inertial stability in the atmosphere, and compare these predictions with observational data, to assess the validity of the models. Much of the necessary work for the QE model is covered by Shutts and Cullen [3], and will not be repeated here.

In the PSG model two approximations are required: firstly that if we project the directions of the wind and the pressure gradient on to the equatorial plane, the rates of change of these directions should be small compared with twice the angular rotation rate of the system. This is a very similar approximation to that made by Hoskins in developing the QE model, where the rate of change of direction must be small compared with the Coriolis parameter. Secondly, for PSG, we neglect the term in the kinetic energy arising from the component of velocity parallel to the axis of rotation of the system. One visualisation of this is to imagine that each hemisphere is squashed directly downwards on to the equatorial plane, and that the flow in this plane represents the global flow. It is this approximation which can be used to classify the model: the primitive equations neglect the term in the kinetic energy arising from motion along the local vertical (the hydrostatic assumption). Since vertical speeds are usually less than horizontal ones, the term neglected in the primitive equations is smaller than that neglected in PSG. At the other extreme, the quasi-geostrophic equations neglect all the kinetic energy.

Near the poles, this second PSG approximation is obviously reasonable: the neglected kinetic energy term is associated with a velocity component which is close to the local vertical. The assumption that this term is small is con-

tained in the hydrostatic approximation made in the primitive equations. It is not so clear however, that it is sensible to neglect what we might call the "axial" term in the kinetic energy near the equator. Some justification can be given from the fact that the quasi-geostrophic models produce remarkably good results even though all the kinetic energy is removed from the Hamiltonian: in many cases the kinetic energy is small compared with the potential energy, which allows this approximation to be made.

2 Coordinate systems for the planetary semi-geostrophic equations

While the PSG equations can be expressed in spherical polar coordinates, this is not the system in which they have their simplest form. Shutts begins by working in a Cartesian (x, y, z) system, centred at the centre of the planet, with the z axis along the axis of rotation, so that the rotation vector is $(0, 0, \Omega)$. This system is then transformed to a hybrid coordinate system by replacing the z coordinate by a pressure dependent coordinate Z. We then have

$$X = x, \tag{1}$$
$$Y = y, \tag{2}$$
$$Z = Z_a \left\{ 1 - (p/p_*)^{(\gamma-1)/\gamma} \right\}, \tag{3}$$

where γ is the ratio of specific heats (C_p/C_v), p_* is taken to be the mean sea-level pressure, and

$$Z_a = \frac{\gamma p_* \alpha_*}{(\gamma - 1)g}, \tag{4}$$

where g is the acceleration due to gravity and α_* is the mean specific volume at pressure p_*. This local vertical coordinate is evidently zero at the surface of the planet, and increases as we move outwards from the surface. We thus have a coordinate system in which X and Y are Cartesian directions in the equatorial plane, and Z is a radial coordinate.

However, if we wish to use observational data collected to check the accuracy of the approximations made, it is useful to transform the equations to another coordinate system. If we choose some point on the surface of the planet, with longitude ϕ_0 and latitude λ_0, we can construct a plane which is

tangent to the sphere at this point. We then define a right-handed orthogonal coordinate system, (x', y', z') so that the z'-axis is normal to the plane, and points outwards (i.e. at the point of contact, z' is the local vertical height), and x' and y' lie in the plane (i.e. locally horizontal). The x' and y' axes are chosen so that locally x' points eastwards, and y' points northwards.

The coordinates of a point in this new coordinate system can be found from those in the (x, y, z) system by the transformation

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin\lambda_0 & \cos\lambda_0 \\ 0 & -\cos\lambda_0 & \sin\lambda_0 \end{pmatrix} \begin{pmatrix} -\sin\phi_0 & \cos\phi_0 & 0 \\ -\cos\phi_0 & -\sin\phi_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix}, \tag{5}$$

where a is the radius of the earth. The Jacobian of this transformation, $J\left(\frac{x',y',z'}{x,y,z}\right)$, is equal to 1.

3 Inertial stability in the PSG model

3.1 Derivation of the stability conditions

Shutts [2], demonstrates that the potential vorticity, q, can be expressed as a Jacobian

$$q = \frac{1}{\rho'} J\left(\frac{M, N, S}{X, Y, Z}\right), \tag{6}$$

with

$$M = 2\Omega x + v, \tag{7}$$
$$N = 2\Omega y - u, \tag{8}$$
$$S = \theta/\theta_0, \tag{9}$$

where (u, v, w) are the components components of wind speed in the (x, y, z) directions, θ is the potential temperature, $\theta_0 = p_*\alpha_*/R$, and R is the gas constant.

We can re-write the expressions for M and N in (x', y', z') coordinates using the transformation equation (5). Without loss of generality, we can choose the original Cartesian x and y axes such that $\phi_0 = -\pi/2$. This has the advantage that the x' axis becomes parallel to the x axis, simplifying the equations for M and N to

$$M = 2\Omega x' + v' \sin \lambda_0, \tag{10}$$

$$N = 2\Omega \{y' \sin \lambda_0 - (z' + a) \cos \lambda_0\} - u', \tag{11}$$

where (u', v', w') are the components of wind speed in the (x', y', z') directions. These equations also contain the approximation that w' is negligible compared with u' and v'.

In comparisons with observational data, it is most convenient to use equations (7) and (8) with u and v written in terms of u' and v', since the latter pair of variables correspond to the eastwards and northwards components of wind speed that are measured. However, the form in (10) and (11) is useful for comparison with Hoskins' QE model.

From equation (6) for the potential vorticity, we can use the properties of Jacobians to write

$$q = \frac{1}{\rho'} J \left(\frac{M, N, S}{x', y', z'} \right) \frac{\partial z}{\partial Z}. \tag{12}$$

Following Shutts and Cullen [3], the inertial stability condition requires that the potential vorticity is greater than zero. For this to be true, the Jacobian must be positive in the northern hemisphere and negative in the southern. For the Jacobian to be positive, it is a necessary condition that the elements on the leading diagonal must be positive. In particular, $\partial M/\partial x'$ and $\partial N/\partial y'$ must be positive.

However, for the Jacobian to be negative, as required in the southern hemisphere, the product of terms on the leading diagonal must be negative. Since $\partial\theta/\partial z'$, and consequently $\partial S/\partial z'$ must be positive for gravitational stability, we require either

$$\partial M/\partial x' > 0 \quad \text{and} \quad \partial N/\partial y' < 0 \tag{13}$$

or

$$\partial M/\partial x' < 0 \quad \text{and} \quad \partial N/\partial y' > 0. \tag{14}$$

3.2 Comparison of the stability conditions in the QE and PSG models

If we look at the expressions for these derivatives in the PSG and QE models, we get for the PSG model

$$\frac{\partial M}{\partial x'} = f\left(1 + \frac{\sin^2\lambda_0}{f}\frac{\partial v}{\partial x'}\right), \tag{15}$$

$$\frac{\partial N}{\partial y'} = f\sin\lambda_0\left(1 - \frac{1}{f}\frac{\partial u}{\partial y'}\right). \tag{16}$$

where f is the Coriolis parameter, $2\Omega\sin\lambda_0$. For the QE model, we have inequalities which, if we are to have stability, place restrictions on $\partial v/\partial x'$ and $\partial u/\partial y'$ (see [3])

$$f\frac{\partial M}{\partial x'} = f^2\left(1 + \frac{1}{f}\frac{\partial v}{\partial x'}\right) > 0, \tag{17}$$

$$f\frac{\partial N}{\partial y'} = f^2\left(1 - \frac{1}{f}\frac{\partial u}{\partial y'}\right) > 0. \tag{18}$$

The conditions for the QE model indicate that in the southern hemisphere, the inertial stability criteria will be violated if either $\partial v/\partial x'$ is sufficiently large and positive, or $\partial u/\partial y'$ is sufficiently large and negative. (To be precise, $\partial v/\partial x' > |f|$, or $\partial u/\partial y' < f$.) We would expect these qualitative conditions to be the same in the PSG model, although the details may be different. Looking at equations (15) and (16) it is clear that to preserve these qualitative features, we require $\partial M/\partial x' < 0$ in the southern hemisphere.

3.3 Use of observational data for validation of stability criteria

The most useful observations for comparison with the model are the wind speed data from aircraft using the automatic ASDAR recording system. Ideally, we wish the aircraft to be flying in a straight line at constant height: this condition is usually approximately true along part of any aircraft's flight path. We also want the flow to be reasonably parallel, so that we can neglect any contribution to the potential vorticity arising from curvature of the

streamlines. We can then define new axes, (ξ, η) in the x'-y' plane so that the ξ-axis is parallel to the flight path, and the η-axis is perpendicular to it. In the QE model, there are no privileged horizontal directions, and we can freely choose the x' and y' axes to coincide with ξ and η. Equation (17) then gives a very simple condition which can be evaluated directly from the observational data.

However, for the PSG model, this simplification is not possible. It is quite straightforward to evaluate $\partial M/\partial \xi$ and $\partial N/\partial \eta$, for example, but we can not say that one *must* be positive, and the other negative: all we know is that the product of the two must be negative. Since it is not possible to evaluate both derivatives from data obtained along a line, it is not possible to determine stability by this method. We must therefore look for cases in which the angle between the flight path and either the x' or y' axis is small, and look at $\partial M/\partial x'$ or $\partial N/\partial y'$ as appropriate. In fact, when looking for differences between the two models, it is only worth looking at $\partial M/\partial x'$, since the condition $\partial N/\partial y' > 0$ is identical in the two models; the only difference between equations (16) and (18) is a factor of 2Ω which is positive, and so has no effect on the sign. Since $f^2 \geq 0$, and the sign change required between hemispheres for $\partial M/\partial x'$ can be supplied by multiplying equation (15) by f, giving a factor of f^2, we can simplify the requirement on $\partial v/\partial x'$ in each of equations (15) and (17) to:

$$\left(1 + \frac{\sin^2 \lambda_0}{f} \frac{\partial v}{\partial x'}\right) > 0 \qquad \text{(PSG condition)}, \tag{19}$$

$$\left(1 + \frac{1}{f} \frac{\partial v}{\partial x'}\right) > 0 \qquad \text{(QE condition)}. \tag{20}$$

From this it is clear that the PSG condition is always less stringent than the QE condition, since $\sin^2 \lambda_0 < 1$. Therefore, if we have already established that the stability condition for the QE model (equation 19) holds in a particular case, then we can immediately say that the PSG condition also holds. It is only in those cases (if any) where the QE condition does not hold that we need to consider the PSG condition. For convenience when discussing the QE condition later, let us here introduce the variable

$$\beta = \frac{1}{f} \frac{\partial v}{\partial x'}, \tag{21}$$

so that the QE condition can be re-expressed as

$$1 + \beta > 0. \tag{22}$$

Cases where the PSG condition holds while the QE condition does not are likely to be near the equator, where $\sin^2 \lambda_0$ is relatively small. In order to evaluate $\partial v/\partial x'$, we require aircraft which fly more or less along a line of latitude. Unfortunately, there are very few aircraft which do this near the equator, and it is consequently very difficult to find such cases. In fact, in the study described below, which examined all the ASDAR data recorded over 52 six-hour periods, not one case satisfying these requirements, along with the requirement that the flow should remain reasonably parallel, was found. Therefore, most of what follows is a comparison of the QE model with observational data.

4 Data extraction

4.1 Observational data

The observational data required is archived into month long datasets. The first stage in the extraction procedure is to choose which data file and which aircraft callsign we are interested in. Programs such as Bruce Ingleby's `opdplot` can be used to examine the `.opd` files to choose callsigns and domains of interest. The data is then extracted for a prescribed area defined using latitude and longitude. We then convert the latitude and longitude of the observation into cartesian x' and y' coordinates on a plane which is tangent to the surface of the earth. The point of contact is chosen so that it has the average longitude of the furthest east and furthest west observations, and the average latitude of the furthest north and furthest south observations. (With hindsight, it might make more sense to choose the point of contact with the average longitude and latitude of all the observations.) These cartesian coordinates (in units of kilometres) are then stored along with the original latitude and longitude, the components of wind speed, u' and v' (in units of metres per second) along the x' and y' directions, and the temperature. The next task is to choose ξ and η axes so that ξ is along the direction of the flight path. A coordinate transformation from (x', y') coordinates, is now performed given an initial and final observation. Next a best-fit straight line through the x' and y' coordinates of the observations lying between (and including) those specified is calculated, and the direction of this line is taken to be the ξ-axis. Only the straight section of a flight path is chosen where the aircraft is not involved in landing or take-off (when the observations are much closer together than normal). Graphs of ν against ξ, and T against ξ are now plotted. The second graph is most useful for

rejecting sets of observations with sharp changes in temperature, whilst linear regression can again be performed on the upper graph. The procedure, amongst other results, prints out the gradient of the best-fit line chosen, and the gradient divided by a calculated value of f. This latter value is the term appearing in equation (20), the condition for stability in the QE model. The value of f is calculated using the average latitude of those points selected for the linear regression: this is not necessarily the ideal procedure when the data points lie very close to the equator, since f will then vary dramatically over the set of points.

5 Results

The observations looked at were those from the first of each month from July 1992 to June 1993, plus 2nd February 1993. In these data were found 121 cases of aircraft flying along straight courses, through parallel flows, at more or less constant height (i.e. not involved in take-off or landing manœuvres), between latitudes $\pm 30°$. Although the height of the aircraft is not itself recorded, the pressure is, and this should not have any rapid changes in it if the aircraft is flying at constant height. For each case, where the data showed a clear wind speed gradient (i.e. more than three roughly collinear points on a graph of ν against ξ), the derivative $\partial\nu/\partial\xi$ (equivalent to $\partial v/\partial x'$ with a new definition of axes) was evaluated. In only five cases was the QE inequality (20) violated: from inequality (22), this occurs when $\beta < -1$. All five cases were closer to being north-south than east-west, making it impossible to test the PSG model condition on $\partial v/\partial x'$, and only allowing a test of the condition on $\partial u/\partial y'$, which we have established is the same as the condition in the QE model. For this reason, the only conclusions which can be drawn are those concerning the QE model. It is unfortunate that aircraft (at least those with ASDAR equipment) do not tend to fly along east-west courses within the tropics since they provide the only data of high enough resolution to be useful in this study.

Graphs of ν against ξ and T against ξ are presented in figures 1–5; the analyses of the areas surrounding the flights are shown as figures 6–9. The most useful information is the analysis of wind direction, showing where there is curvature in the wind field. The temperature information is not so much use, since the aircraft was unlikely to be flying at the 250mb level, although it will almost certainly be between 200 and 300mb. The difference in wind speeds at the the two levels, however, will be relatively small.

Figures 2, 3 and 5 all show temperature variations of less than 1°C over the flight path. Although figure 1 shows a steady increase in wind speed, by about 3°C over 1200 km, we can still use the data from this case. The temperature change is not in fact caused by the aircraft descending: the pressure readings from the aircraft do remain constant over this part of the flight path. The remaining case, figure 4 has sharp discontinuities in both wind speed and temperature after the first four data points. If the pressure data are examined, we see that the first four observations are all at the same pressure (i.e. more or less the same height), and that the following four observations are all at the same pressure as each other, although this is higher than for the previous four. The aircraft has presumably descended from one level flight path to another, and we can take each set of four observations separately, and look at the wind speed gradient of these.

For the five cases, we can draw best-fit lines through points on the ν-ξ graphs of figures 1–5. Dividing the gradient by the appropriate Coriolis parameter for the average latitude λ_0 of the chosen data points gives values of β as follows.

mmddhh	Callsign	λ_0	β
080112	QF0005	-16.9°	-1.10
100118	QF0000	-4.3°	-1.17
050100	QF0059	-22.1°	-1.15
050100	AN0002	-20.2°	-1.49
050118	QF0050	-19.7°	-1.64

The two cases of most interest are the flights with callsigns AN0002 and QF0050, since these show the most negative values of β. Looking at the analysis for QF0050 (figure 9), there is some curvature of the wind field, and some change of direction of the observations themselves, which reduces the weight we can give to this case. However, β is sufficiently large and negative to cast some doubt on the validity of the QE condition. The analysis for AN0002 is more definitive: there is very little curvature in the analysed field, confirming the validity of the data, and suggesting that the QE condition is not holding. Coupled with the three other cases (where the analyses all show reasonably parallel flow, at least in the vicinity of the observations), this seems to be evidence to cast doubt on the QE condition, and hence on the validity of using the QE model in the tropical region.

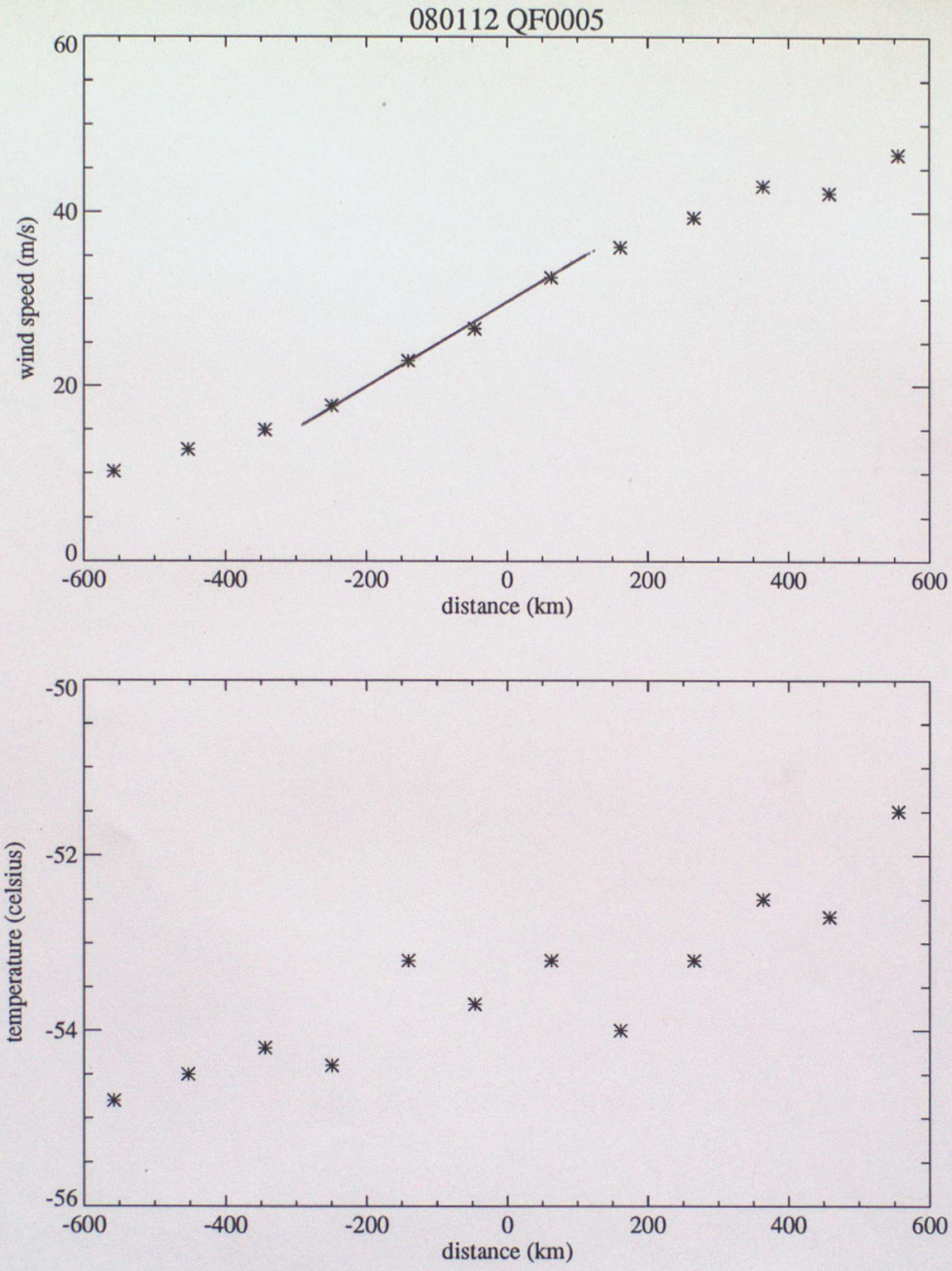

Figure 1: Observations for callsign QF0005

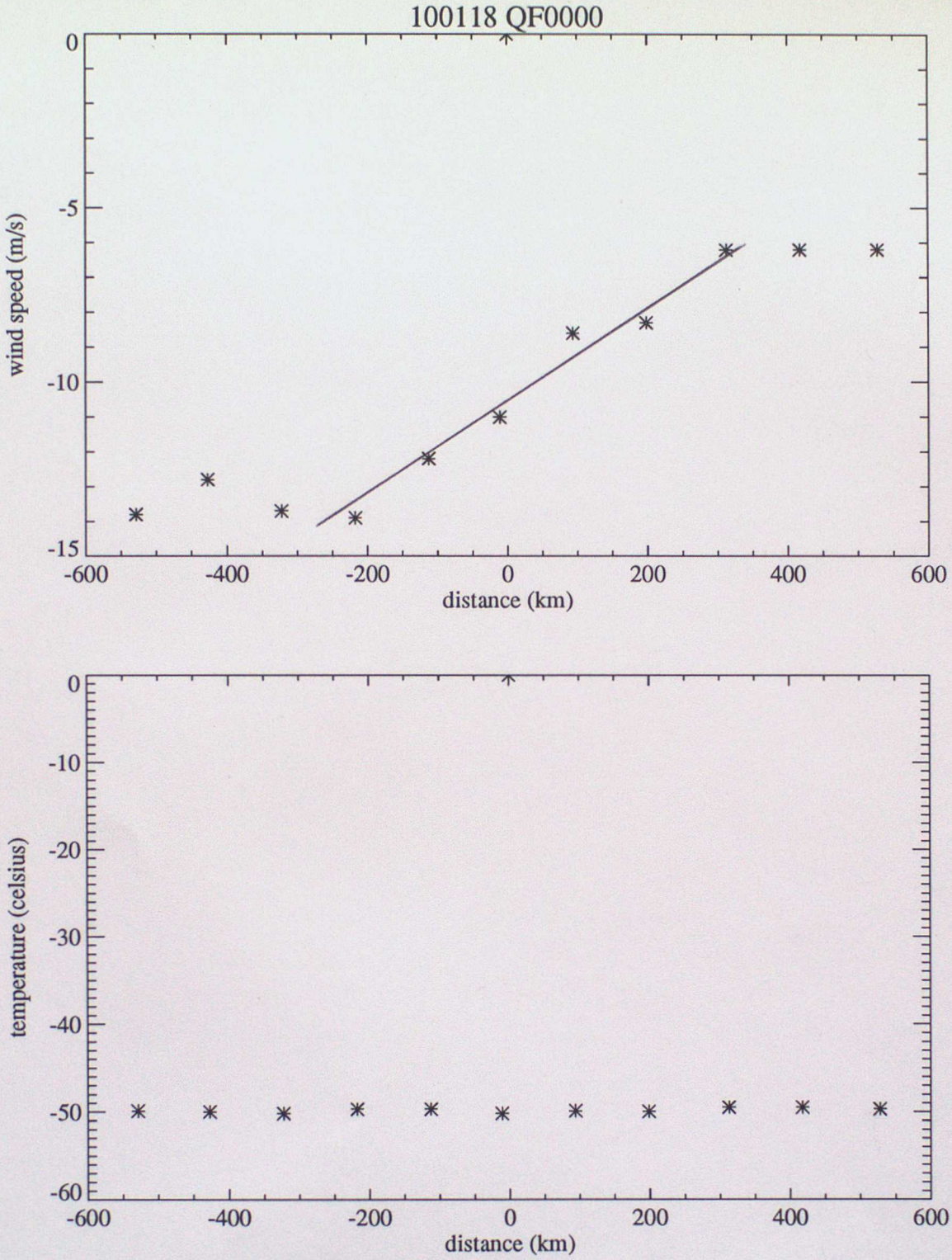

Figure 2: Observations for callsign QF0000

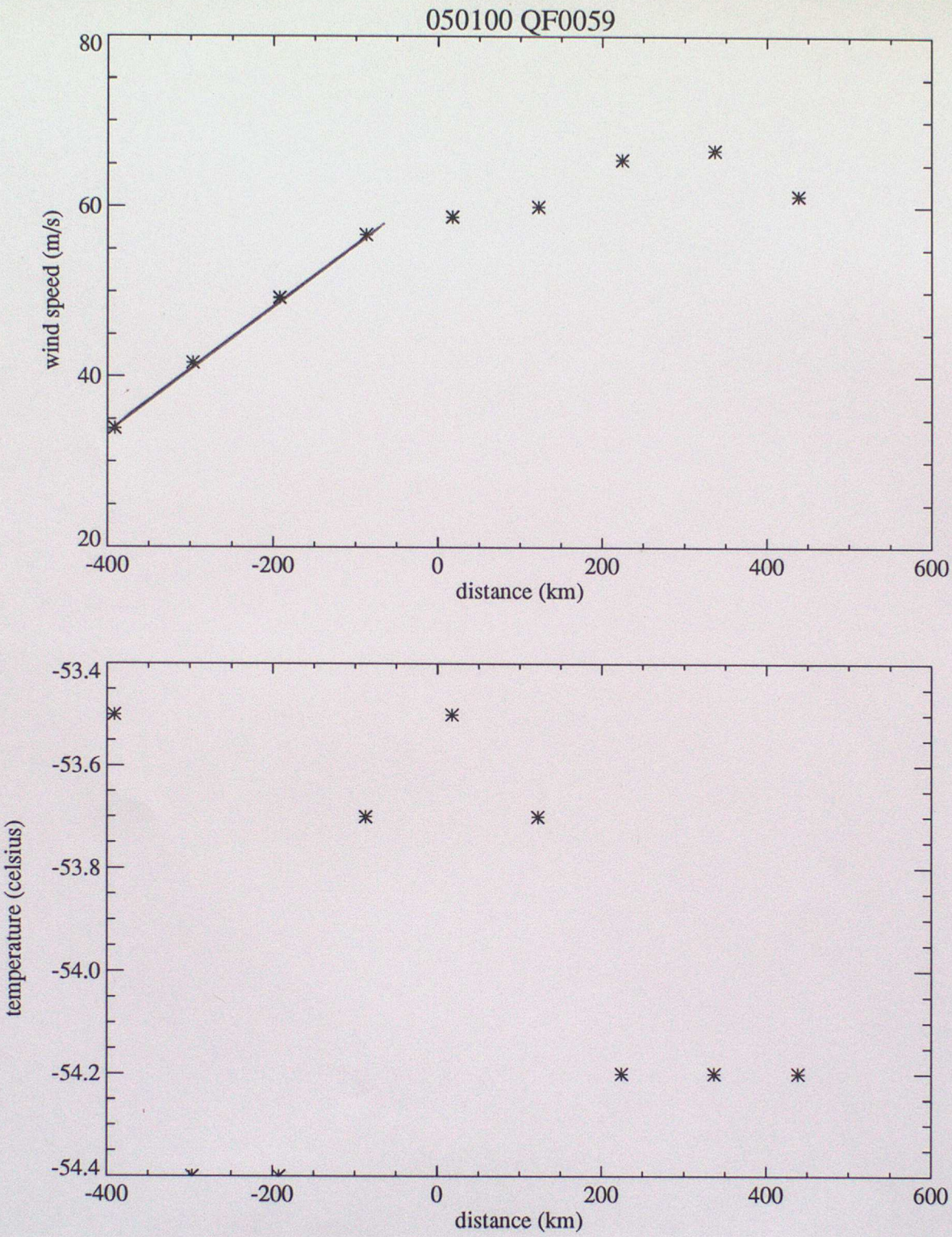

Figure 3: Observations for callsign QF0059

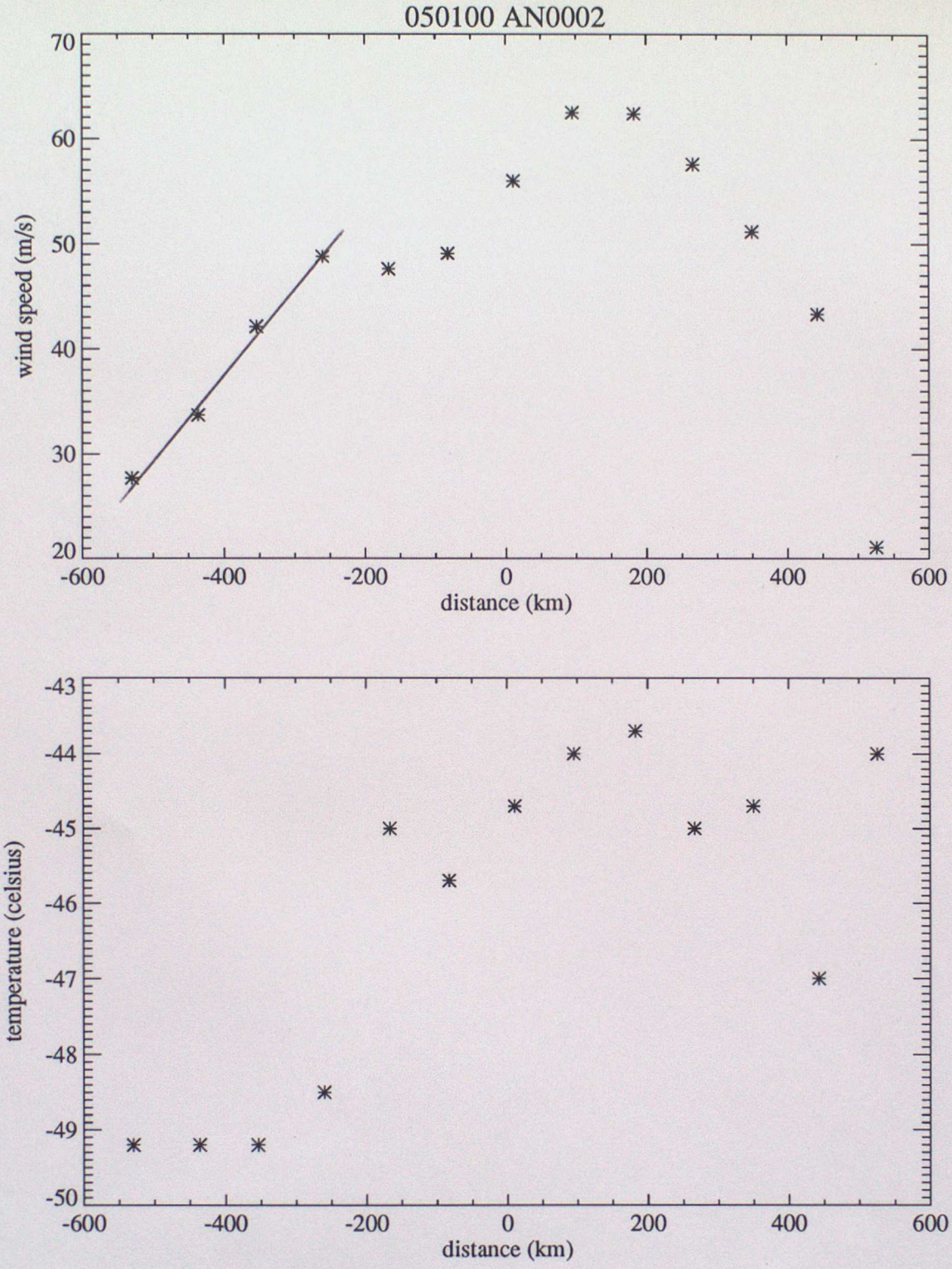

Figure 4: Observations for callsign AN0002

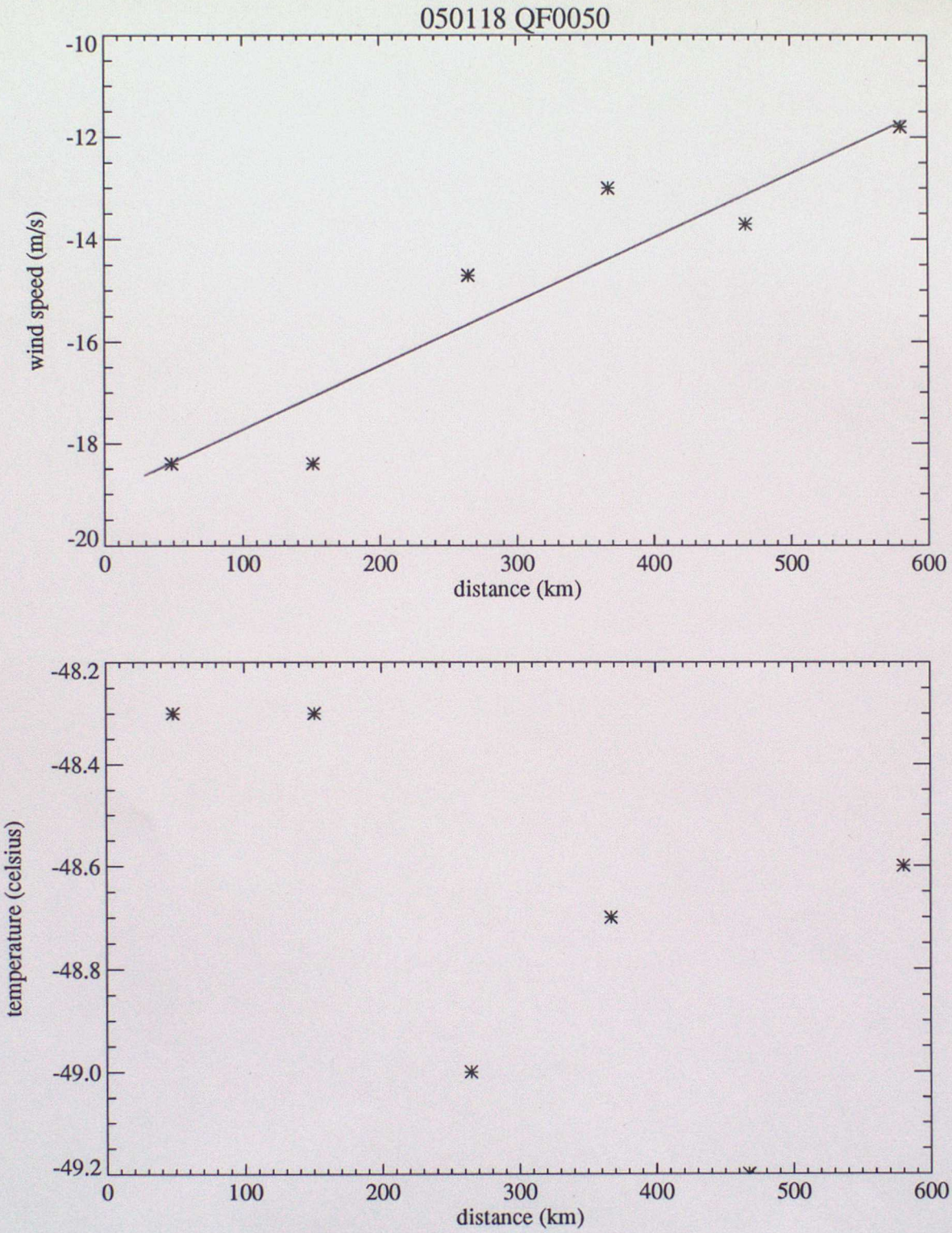

Figure 5: Observations for callsign QF0050

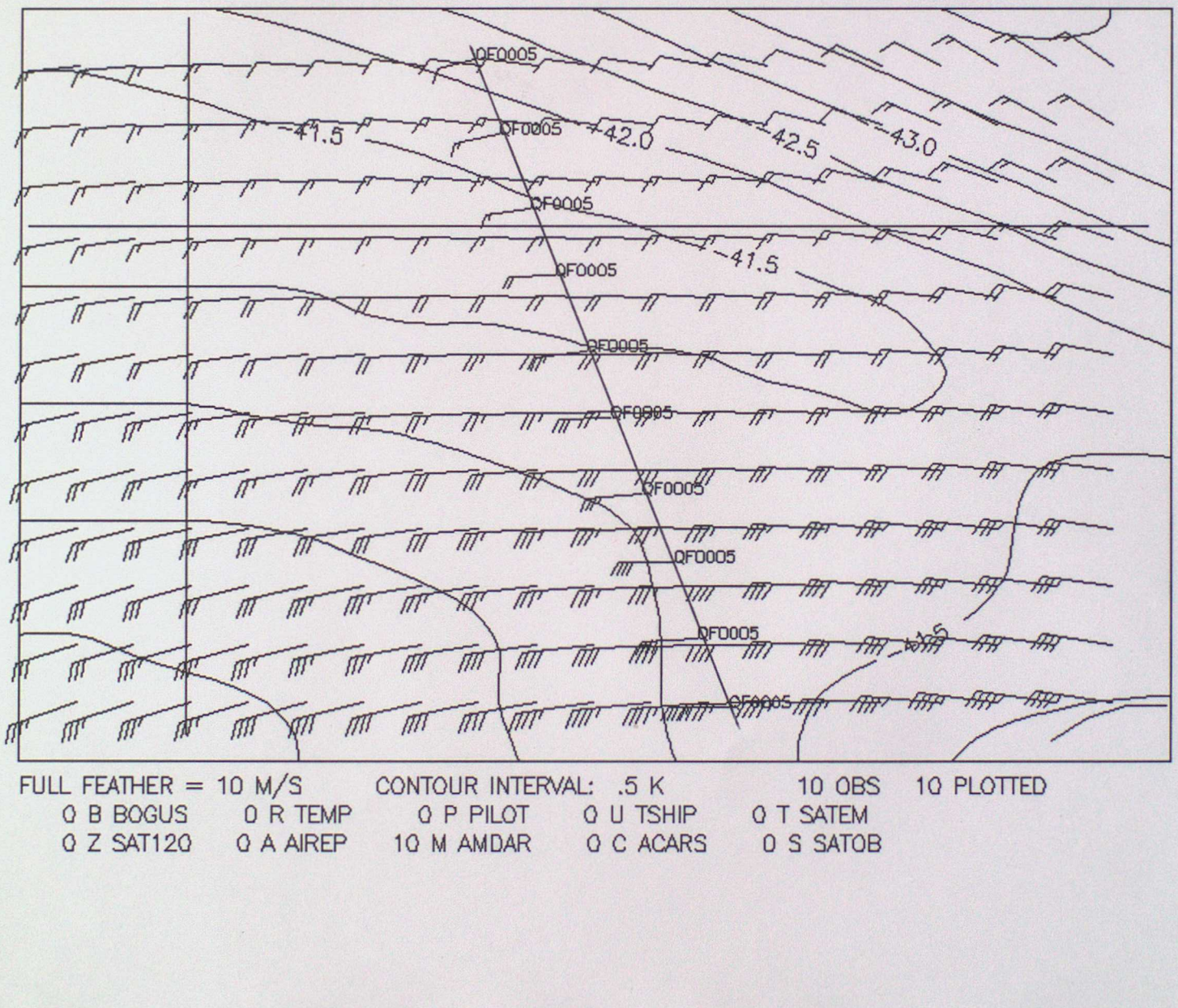

Figure 6: Analysis for callsign QF0005

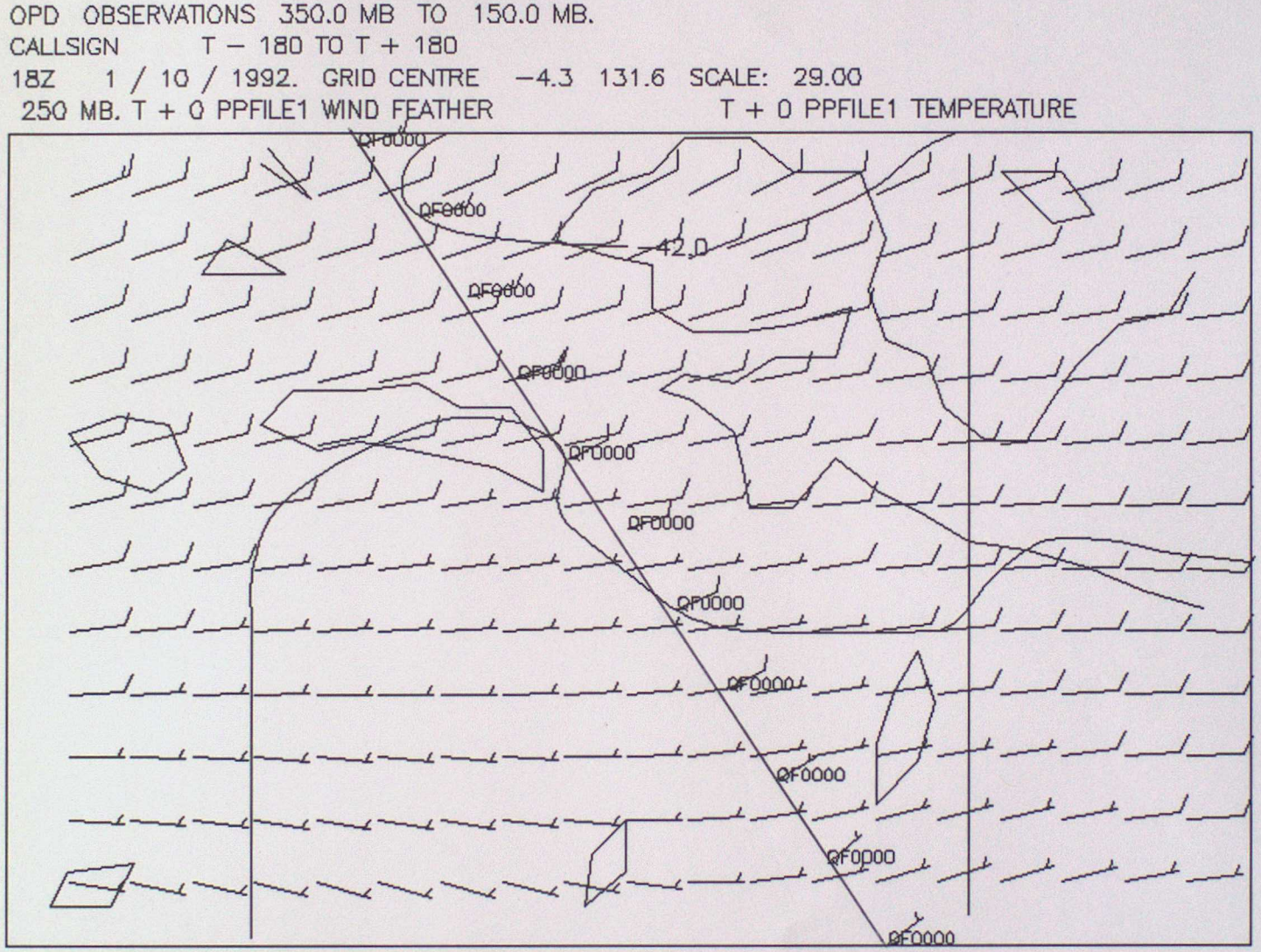

Figure 7: Analysis for callsign QF0000

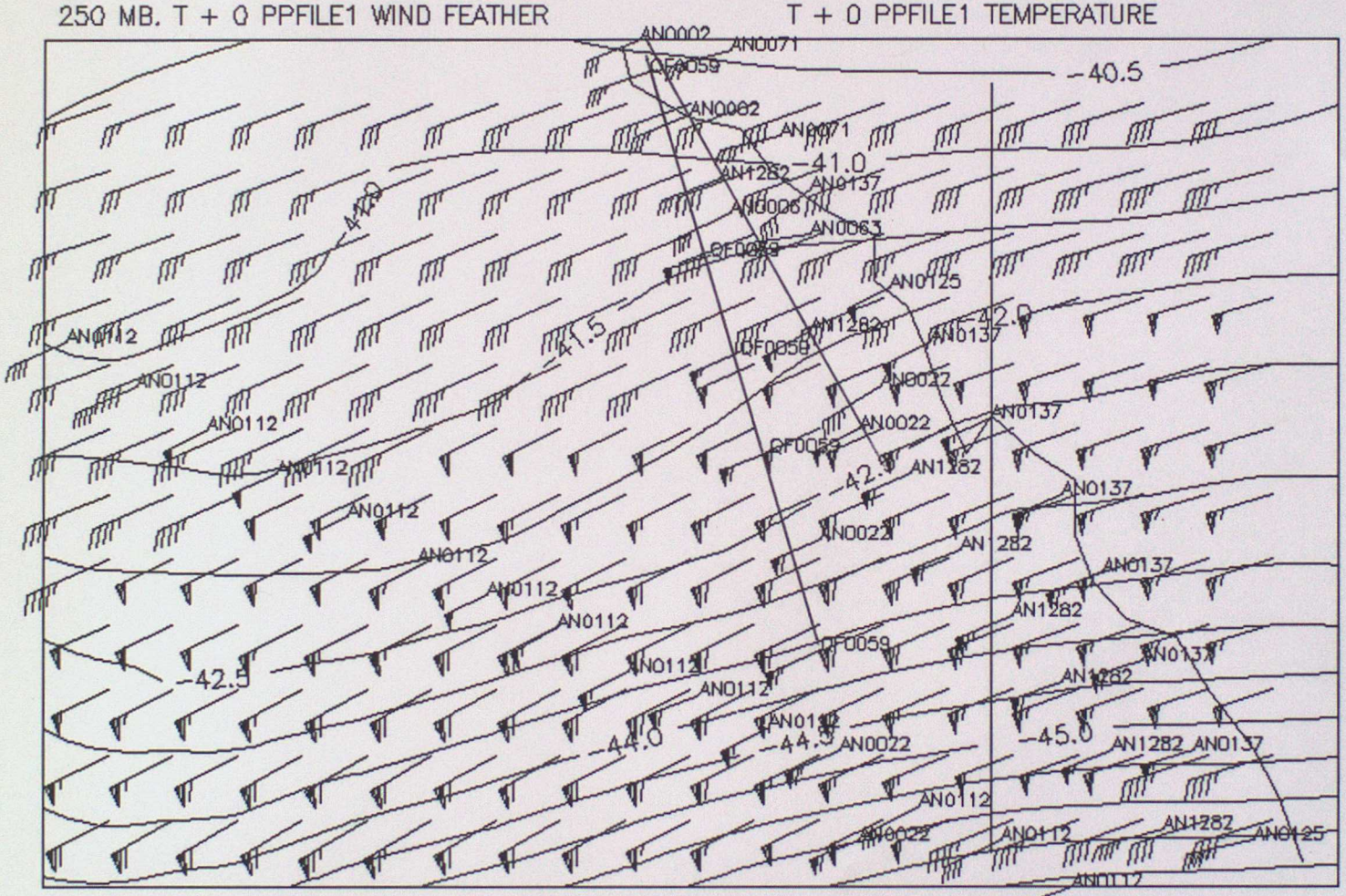

Figure 8: Analysis for callsigns QF0059 and AN0002

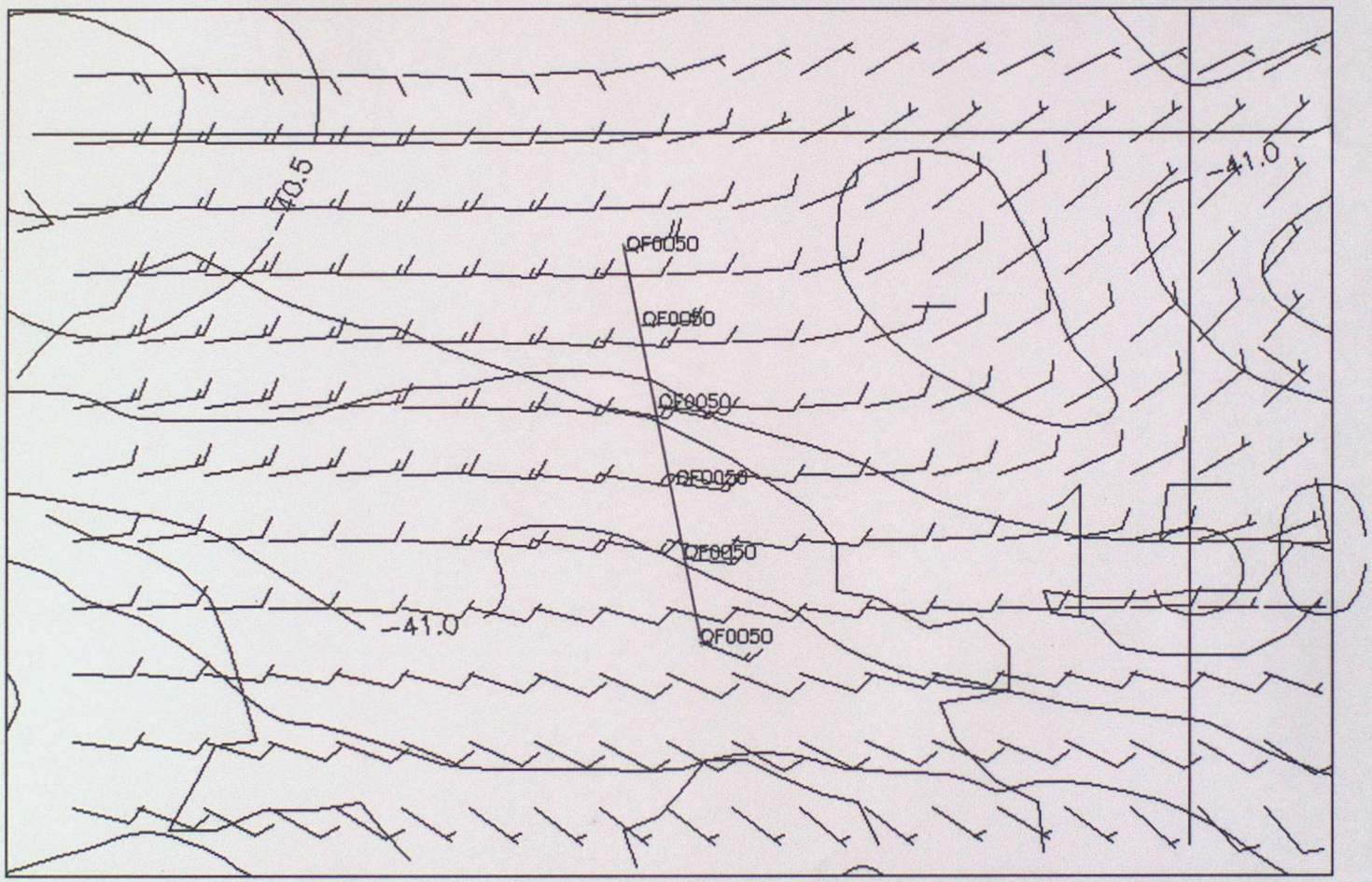

Figure 9: Analysis for callsign QF0050

5.1 Scatter of β-values

One other test of the QE condition is to see if there is a higher than expected number of cases where the value of β is just above -1. If there is, then this would suggest that there is some cut-off at -1 being enforced, and that the QE equations and the QE condition are sensible descriptions of the atmosphere. Figures 10 and 11 show frequency histograms of beta, with frequency on the vertical axis, and β horizontally. Figure 10 divides the β axis into intervals of 0.1; figure 11 divides it into intervals of 0.2. (For readers who wonder why there is such a preponderance of negative β-values, it should be pointed out that the vast majority of the data came from the southern hemisphere. This bias is due to the fact that most of the aircraft equipped with ASDAR equipment fly on routes either within Australasia, or between Australasia and south-east Asia.) Neither plot shows any clustering around -1; the data appear to tail off smoothly at large negative values of β. Admittedly, we have only a small number of observations in the tail: some anomalous behaviour which is missed here might appear in a larger study. However the data would seem to show that there is nothing special about the value $\beta = -1$, and that larger negative values of β occur quite freely (although not very frequently) in the atmosphere.

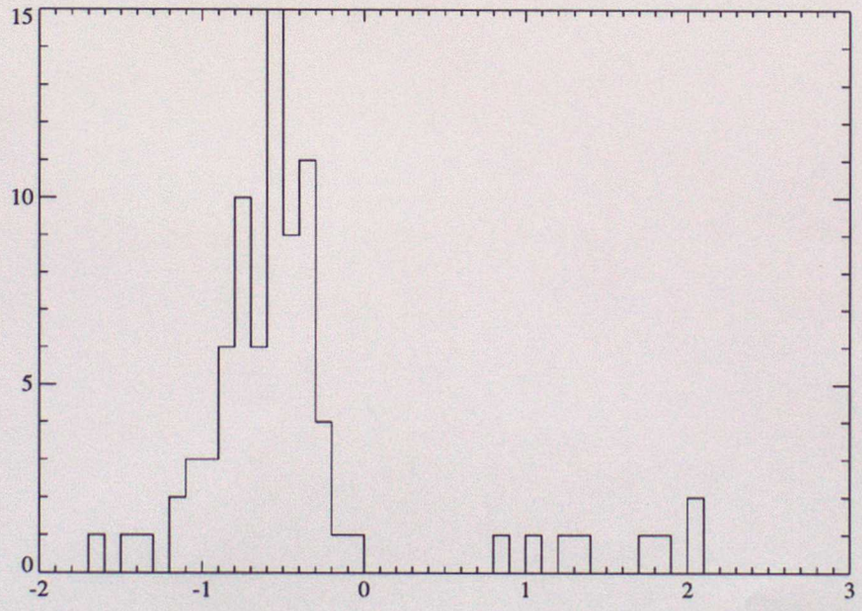

Figure 10: Frequency histogram of β - interval 0.1

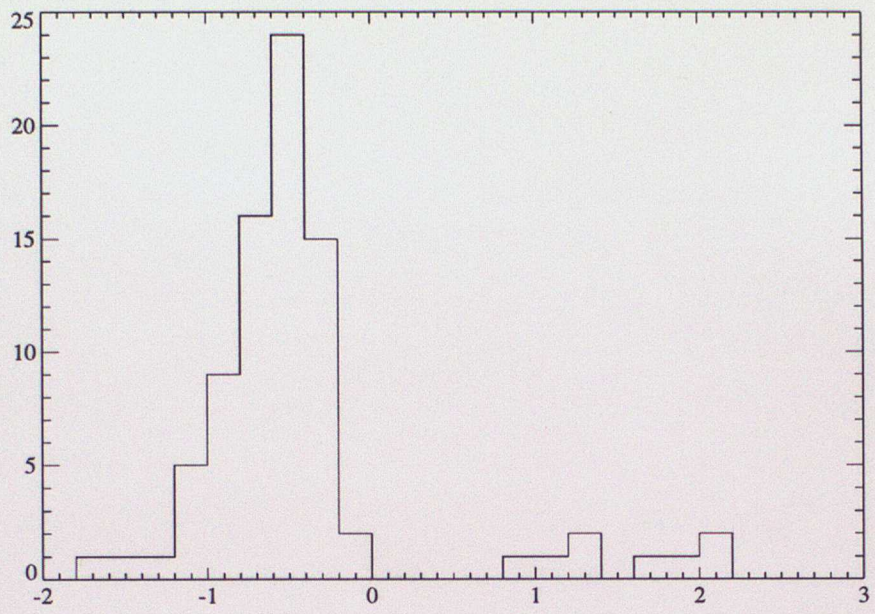

Figure 11: Frequency histogram of β - interval 0.2

6 Conclusion

In conclusion, the study of the inertial stability conditions for the QE and PSG models suffers greatly from lack of suitable observational data: the data useful for this study is a small fraction of that collected. From the limited data available here, we have seen that in general the tropical atmosphere is inertially stable. However, there are a few cases where the data collected shows flows in the atmosphere which are unstable and could not be represented by the QE model. A larger scale study would give a more definitive conclusion, but this would require the data extraction and analysis procedure to be largely automated. This does not seem to be impossible: we would require routines to assess "straightness" and "levelness" of flight path, "straightness" of air flow across the flight path, and an automated method for assessing gradients of sections of the graph of wind-speed component against distance. Although this is not impossible, it would seem to involve a fair amount of work.

It has proved impossible to find a simple way to use the available data to test the inertial stability condition for the PSG model. To do so ideally requires data along an east-west line, which within the tropics does not often

occur. However, it might be possible to use the data from aircraft flying along parallel paths (which does happen, particularly over Australisia) to evaluate both of the factors $\partial M/\partial \xi$ and $\partial N/\partial \eta$ which appear in the expression for the potential vorticity, and hence determine stability. However, this is likely to involve searching for a needle in a haystack: the number of cases where sufficiently good data can be obtained will be very small. Short of commissioning an aircraft to fly along the required flight path, it is difficult to see how else to obtain the necessary data. It would seem therefore that the validity of the PSG model can most easily be determined by seeing how well it models features of the atmosphere, such as the various types of wave which propagate near the equator.

7 Acknowledgements

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